THE GEOMETRY OF CUP-AND-RING MARKS

By A. Thom

THIS paper is based entirely on a set of rubbings kindly made available to the author by R. W. B. Morris. A knowledge that the unit of length used in setting out these marks is exactly one-fortieth of the Megalithic yard enables the geometry to be unravelled. This geometry and that which we find in the Megalithic stone circles are both controlled by the same rules and conventions.

It has been shown in Thom (1968) that a definite unit of length appears in the diameters of the rings associated with cup-and-ring marks when these rings are intended to be true circles. This unit will here be called the Megalithic inch "(mi.)". Its value is 0·816 British Standard inches (20·73 mm.) or exactly one-fortieth of the Megalithic yard (2·720 ± 0·003 ft.). It has also been shown that when the marks are not pure circles the geometry follows generally the same rules as apply for the stone rings (Thom, 1967). These rules are:

1. Any length used in the construction is to be an integral multiple of the unit.
2. The perimeter of any ring used is to be as near as possible an integral multiple of $2\frac{1}{2}$ primary units.

The second rule may have arisen from the fact that the circumference of a circle of diameter 8 is very nearly 25.

It has been shown (Thom, 1967) that the primary unit may be subdivided into halves or quarters but never into thirds. For longer distances $2\frac{1}{2}$, 5 or 10 primary units were frequently used.

There is evidence that sometimes the rock surface was prepared by being rubbed and polished to a smooth if not always a plane surface. It has become evident from the geometry that the designs must have been set out originally with a precision approaching that attained today by a mechanic using a finely-divided scale, a scriber and dividers. Unless archeologists produce evidence to the contrary we can rule out dividers and assume that beam compasses or trammels were used and that these would not be adjustable. The Megalithic draughtsman would use a set of trammels with the distances between the scribing points (flint or quartz?) advancing by Megalithic inches or perhaps half and quarter inches. Thus no divided scale such as we use today would be necessary. The trammels for 1, 2, 5 and 8 mi were easily checked by stepping 40, 20, 8 and 5 times along the standard yard of 40 mi. The other sizes would follow by addition or subtraction. An accuracy of a few thousandths of an inch is possible and if anyone cares to reconstruct the figures shown here he will find that unless this kind of accuracy is maintained the design will get out of hand.

This use of standard trammels was undoubtedly the reason for rule (1) above. Special trammels were precluded or perhaps forbidden and so all construction lengths and the radii of all arcs had to be integral.

An elementary example is found in the spiral at Hawthornden (Fig. 1) which is built up from semi-circles of radii 4, $3\frac{1}{4}$, $2\frac{1}{2}$, $1\frac{3}{4}$ and 1 mi., the common interval in the radii being $\frac{3}{4}$ mi. But we must face up to the fact that in the rock markings as in the stone circles (Thom, 1967) ellipses are frequently found. Perhaps the most interesting and beautiful example is that found at Knock, near Whithorn, Wigtonshire (see Vol. 14, p. 100 (no. 518) of these *Transactions*). Fig 2 shows that this example consists of a spiral built up from 6 half-ellipses and a semi-circle. To understand how remarkable this design really is we must recall the following theorem: Let a be the major axis of an ellipse, b the minor axis and c the distance between the foci. Then always

$$a^2 = b^2 + c^2$$

This is the Pythagorean relation between the hypotenuse and the sides of a right-angled triangle. It follows that if the Megalithic draughtsman followed the first rule when drawing ellipses he had to find for each ellipse a triangle that was the correct size *and* would at the same time satisfy the Pythagorean relation in integers. Let us see how nearly he succeeded in this apparently impossible task.

In Table 1 the first three columns contain the nominal values of a, b and c for the six ellipses. The next column is the calculated value of a assuming that b and c retain their nominal values. The last two columns give the discrepancy in Megalithic inches and in British Standard inches.

TABLE I

a mi.	*b* mi.	*c* mi.	*a* cal.	Discrepancy mi.	Discrepancy inches
$7\frac{1}{2}$	$6\frac{1}{2}$	$3\frac{3}{4}$	7·504	0·004	0·003
$6\frac{1}{2}$	6	$2\frac{1}{2}$	6·500	0·000	0·000
$5\frac{1}{2}$	$4\frac{3}{4}$	$2\frac{3}{4}$	5·489	0·011	0·009
$4\frac{1}{2}$	$4\frac{1}{4}$	$1\frac{1}{2}$	4·507	0·007	0·006
$3\frac{1}{2}$	3	$1\frac{3}{4}$	3·473	0·027	0·022
$1\frac{3}{4}$	$1\frac{1}{2}$	$\frac{7}{8}$	1·737	0·013	0·011

Another remarkable feature of the resulting design is that the spacing of the large whorls on the major axis is everywhere exactly one unit and on the minor axes $\frac{7}{8}$ unit and yet every one of the ellipses is based on an almost perfect triangle. In Fig. 1 we see the design very carefully drawn and superimposed on a rubbing by Mr. R. W. B. Morris of the rock at Knock.

It is almost inconceivable that the accuracy shown in Table 1 could have been obtained graphically while working on a rock surface. How then were these triangles discovered? It seems almost certain that the designer knew the Pythagorean Theorem and could use it to check any assumed triangle. How else did he know that the 12, 35, 37 triangle was exact? He certainly used the theorem at his obviously important site at Woodhenge, and elsewhere (Thom, 1967). The calculations would have been done in units or in quarter-units. It would be convenient if today we had names for his quarter-yard (0·68 ft.), and for his quarter-inch (0·204 in.).

At Hawthornden, Midlothian, there is a more complicated, if less pleasing, design (Fig. 3). This contains a looped spiral built up round two 3, 4, 5 triangles. The larger triangle E B D has sides 3, 4 and 5 mi. The smaller has $1\frac{1}{2}$, 2 and $2\frac{1}{2}$ mi. The centres of all the circular arcs forming the spiral lie on one or other of the five points A, B, C, D and E, as will be obvious from the figure. Starting at G all radii are integral in $\frac{1}{4}$ mi. units until the centre is reached. Coming out $\frac{1}{2}$ units were used. This kind of sequence is, of course, possible only when the foundation triangles have integral sides.

Leaving the spiral by the dotted line centred on G we are led into a short serpentine of which the first semi-circle is centred at F. This point F is placed in a peculiar position such that its distance from each of the centres is very close to an integral number of half units. To prove this, specific co-ordinates were assumed for F, when a little trigonometry gave the following values:

TABLE 2

FG = 17·500	FD = 9·463
FA = 12·514	FC = 13·105
FB = 11·503	FE = 13·945

It will be realized that it is quite impossible to find a point to satisfy more than two of the integral conditions. The first three are so nearly $17\frac{1}{2}$, $12\frac{1}{2}$ and $11\frac{1}{2}$ that it seems likely that the whole design was built up round the discovery that there was a point F related to G, A and B in this way. The designer would have known that the last three are only approximations to $9\frac{1}{2}$, 13 and 14.

Another interesting spiral at Blackshaw, West Kilbride, is shown in Fig 4. This is built up of seven half-ellipses, the particulars of which are given in Table 3.

TABLE 3

a	b	c	$\sqrt{(b^2 + c^2)}$
$12\frac{1}{2}$	11	6	12·530
$10\frac{3}{4}$	$9\frac{3}{4}$	$4\frac{1}{2}$	10·738
9	8	4	8·944
$7\frac{1}{2}$	$6\frac{1}{2}$	$3\frac{3}{4}$	7·504
6	$4\frac{1}{2}$	4	6·021
$4\frac{1}{2}$	3	$(3\frac{3}{8})$	4·516
$2\frac{1}{2}$	$1\frac{1}{2}$	2	2·500

Here the designer has not been so successful with his sizes as at Knock but nevertheless he has found a remarkable set of ellipses. For the last but one I have assumed he used $3\frac{3}{8}$ for the distance between foci but he may well have used $3\frac{1}{2}$ and tolerated the consequent slight reduction in *b*. The $10\frac{3}{4} \times 9\frac{3}{4}$ half-ellipse has been shown dotted because very little trace of it remains on the rock, but since the Pythagorean condition is so nearly fulfilled there is little doubt that the line dotted was the intention.

In looking at this design it must be borne in mind that the designer was striving to satisfy two irreconcilable conditions: To find ellipses which would nest and at the same time satisfy the Pythagorean condition.

In spite of its simplicity the group of 5 "cups" on the living rock on the golf course above Gourock is not without interest (Fig. 5). The five superimposed rings are drawn exactly on the corners of a 3, 4, 5 and a 6, 8, 10 triangle. It will be seen, in spite of the weathering which has taken place, how accurately they fit. And we can go a little further. A ring roughly the size of the cups was drawn on transparent material and moved about until it was judged to be central on a cup. The centre was pricked through to the rubbing and the six distances so found were carefully measured and found to be 2·29, 2·46, 3·12, 3·43, 4·12 and 8·10 in. There are various ways of finding a quantum from such measurements. The weakest is to add them together and divide by the sum of the nominal distances. I added the numbers successively to the first and analysed the sequence so obtained by Broadbent's method. The result was:

$$1 \text{ mi.} = 0{\cdot}815 \pm 0{\cdot}004 \text{ inches}$$

The value of the standard error indicates that it is partly by chance that this result is so near the known value 0·816. Since the standard error of the Megalithic yard is $\pm$ 0·003 ft. or 0·036 in. that of the Megalithic inch is $\pm 0{\cdot}036/\sqrt{40}$ or $\pm$ 0·006 inches.

In conclusion, every legible design so far examined has been found to consist of a geometrical figure set out with a unit of 0·816 inches. When the design is other than a simple circle it is always based on an integral right-angled triangle (or triangles) so that all the leading dimensions are integral. Since the unit is *exactly* one-fortieth of the Megalithic yard, and since the conventions are identical with those governing the design of the stone rings of the Megalithic people, the conclusion is inescapable that the latter were also responsible for the cup-and-ring marks.

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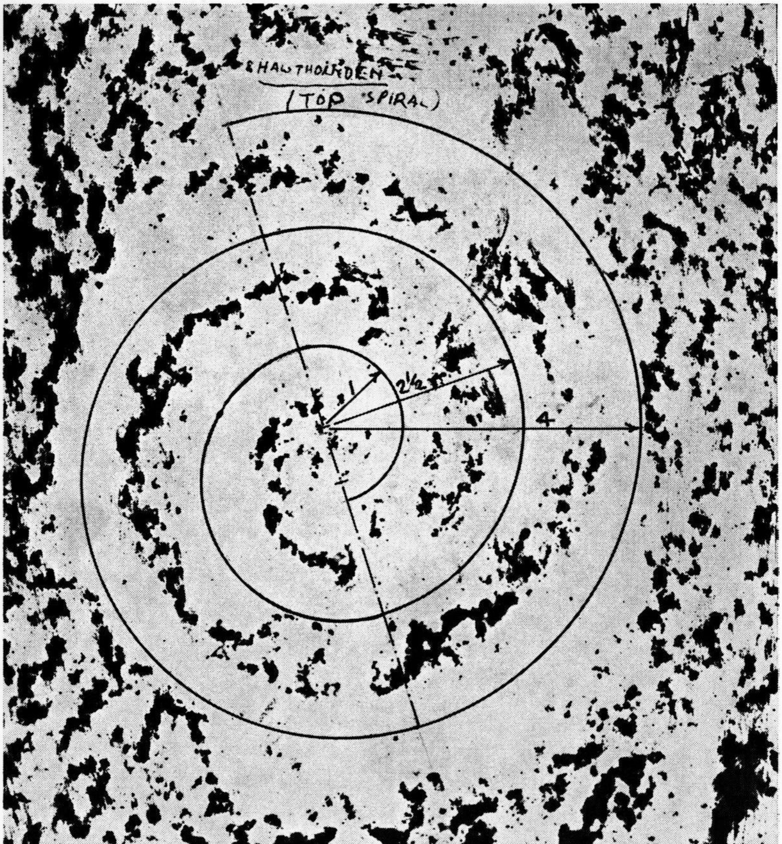

FIG. 1. Hawthornden, Midlothian: the top spiral. [National grid reference: NT 281633.] *Reduced to approximately half size.*

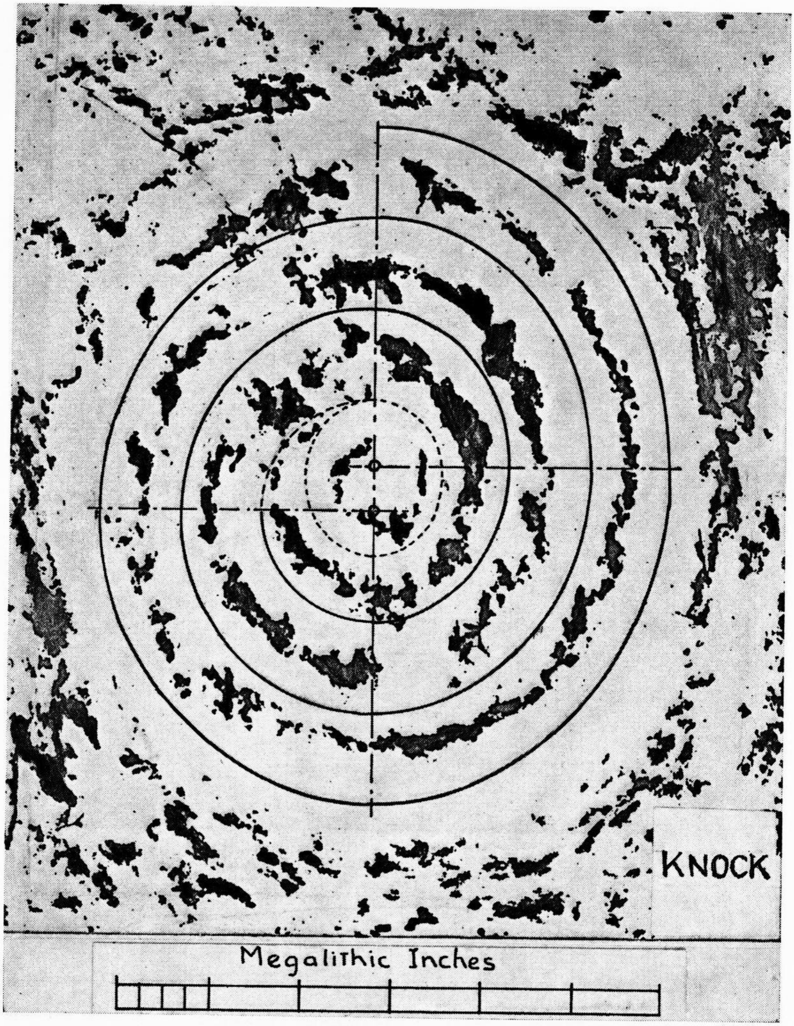

FIG. 2. Knock, Whithorn. [National grid reference: NX 364402.] *Reduced to four-sevenths size.*

Ellipses	a	b	c	M $(b^2 + c^2)$
	$7\frac{1}{2}$	$6\frac{1}{2}$	$3\frac{3}{4}$	7·505
	$6\frac{1}{2}$	6	$2\frac{1}{2}$	6·500
	$5\frac{1}{2}$	$4\frac{3}{4}$	$2\frac{3}{4}$	5·489
	$4\frac{1}{2}$	$4\frac{1}{4}$	$1\frac{1}{2}$	4·507
	$3\frac{1}{2}$	3	$1\frac{3}{4}$	3·473
	$1\frac{3}{4}$	$1\frac{1}{2}$	$\frac{7}{8}$	1·737

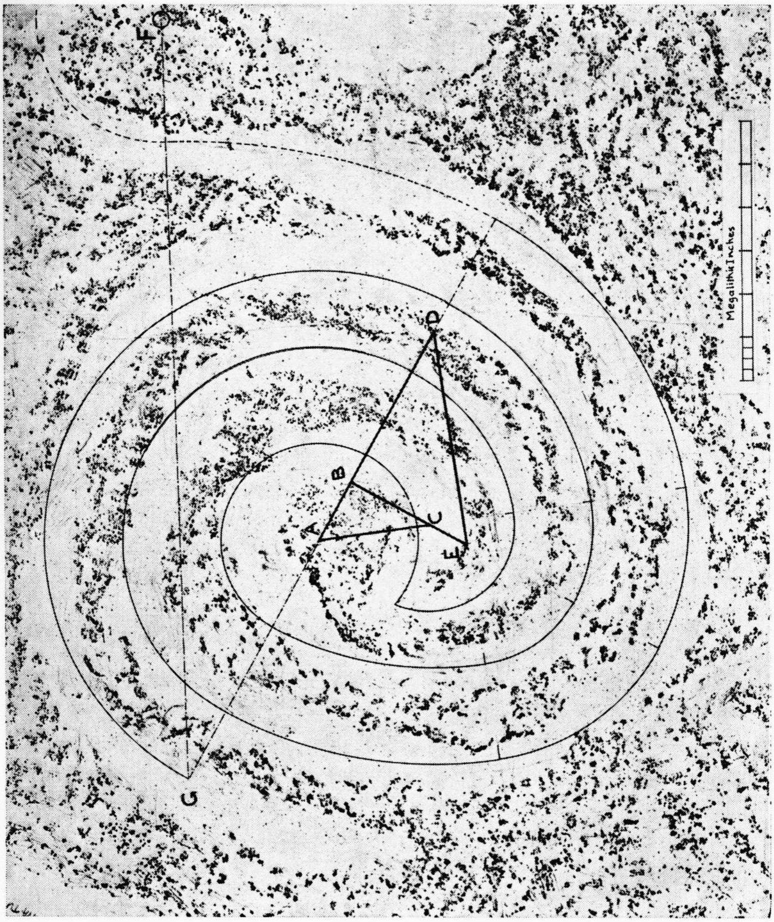

FIG. 3. Hawthornden, Midlothian: the lower central spiral. [National grid reference: NT 281633.]
Reduced to approximately two-sevenths size.

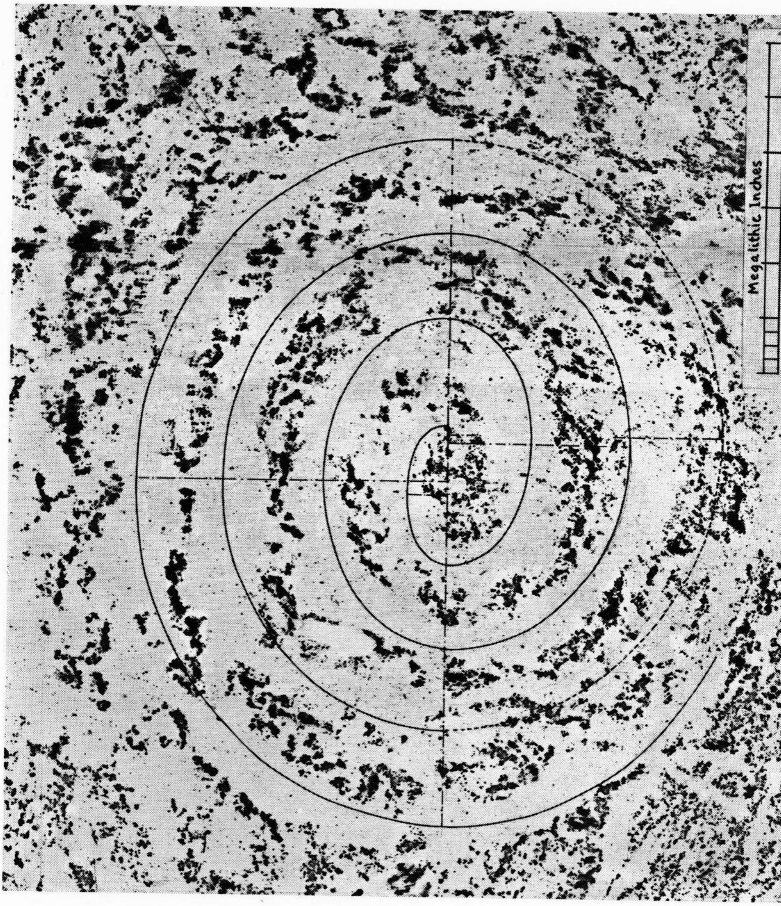

FIG. 4. Blackshaw, near West Kilbride, Ayrshire: the largest spiral. [National grid reference: NS 231484.] *Reduced to approximately four-sevenths size.*

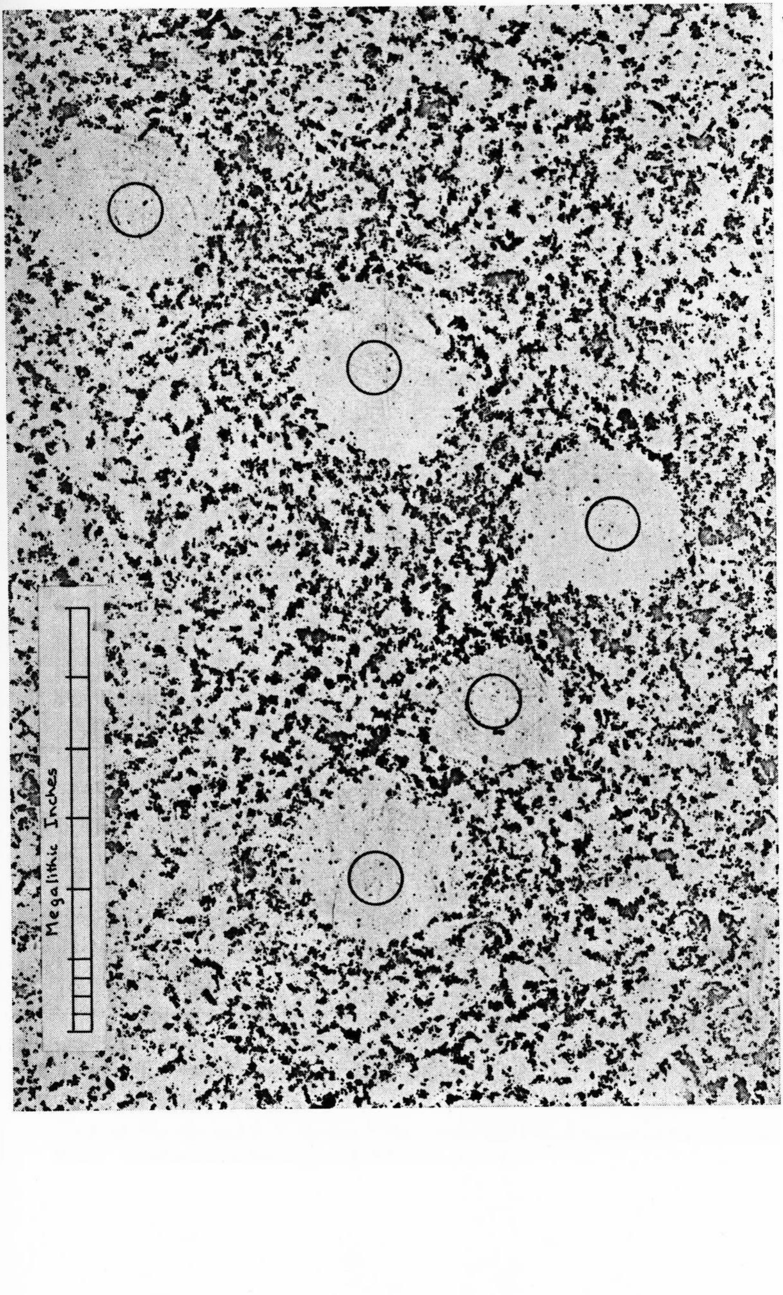

FIG. 5. Gourock Golf Course, Renfrewshire: cup marks. [National grid reference: NS 229762.] *Reduced to approximately four-ninths size.*